Solutions Resit Exam - Statistics 2019/2020

SOLUTION 1:

(a) Let n_j denote the no. of realisations with $X = j$ $(j = 0, \ldots, 3)$. The likelihood is:

$$L(\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \prod_{j=0}^{3} p(j|\theta)^{n_j} = \left(\frac{2\theta}{3}\right)^{n_0} \cdot \left(\frac{\theta}{3}\right)^{n_1} \cdot \left(\frac{2(1-\theta)}{3}\right)^{n_2} \cdot \left(\frac{1-\theta}{3}\right)^{n_3}$$

Switching to the log likelihood:

$$\begin{aligned} l(\theta) = \log(L(\theta)) &= n_0 \log\left(\frac{2\theta}{3}\right) + n_1 \log\left(\frac{\theta}{3}\right) + n_2 \log\left(\frac{2(1-\theta)}{3}\right) + n_3 \log\left(\frac{1-\theta}{3}\right) \\ &= n_0 \log(2\theta) + n_1 \log(\theta) + n_2 \log(2-2\theta) + n_3 \log(1-\theta) - n\log(3) \end{aligned}$$

Taking the derivative of $l(\theta)$ and setting it to zero:

$$\begin{aligned} \frac{d}{d\theta}\, l(\theta) = \frac{2n_0}{2\theta} + \frac{n_1}{\theta} - \frac{2n_2}{2-2\theta} - \frac{n_3}{1-\theta} &= 0 \\ \Leftrightarrow \frac{n_0}{\theta} + \frac{n_1}{\theta} - \frac{n_2}{1-\theta} - \frac{n_3}{1-\theta} &= 0 \end{aligned}$$

yields

$$n_0(1-\theta) + n_1(1-\theta) - n_2\theta - n_3\theta = 0 \Leftrightarrow \theta = \frac{n_0+n_1}{n_0+n_1+n_2+n_3} = \frac{n_0+n_1}{n}$$

To check whether this is really a maximum, we compute the 2nd derivative:

$$\frac{d^2}{d\theta^2}\, l(\theta) = -\frac{n_0}{\theta^2} - \frac{n_1}{\theta^2} - \frac{n_2}{(1-\theta)^2} - \frac{n_3}{(1-\theta)^2}$$

As the 2nd derivative is negative for all θ, we have a maximum.
For the given data ($n_0 = 2$, $n_1 = 3$, $n = 10$): $\hat{\theta}_{ML} = 0.5$.

(b) Compute the expectation of X:

$$E_\theta[X] = \sum_{i=0}^{3} x_i \cdot p(x_i|\theta) = 0 \cdot \frac{2\theta}{3} + 1 \cdot \frac{\theta}{3} + 2 \cdot \frac{2(1-\theta)}{3} + 3 \cdot \frac{1-\theta}{3} = \frac{7-6\theta}{3}$$

Then set $E_\theta[X] = \bar{X}$ and solve for θ:

$$\frac{7-6\theta}{3} = \bar{X} \Leftrightarrow \theta = \frac{7}{6} - \frac{1}{2}\bar{X}$$

For the given data ($\bar{x} = 1.5$): $\hat{\theta}_{MOM} = \frac{5}{12}$.

(c) Note that for a sample of size n: $E_\theta[n_j] = n \cdot p(j|\theta)$. This yields

$$E_\theta[n_0] = n \cdot \frac{2\theta}{3} \text{ and } E_\theta[n_1] = n \cdot \frac{\theta}{3}$$

The ML estimator is unbiased, as

$$E_\theta[\hat{\theta}_{ML}] = \frac{E_\theta[n_0] + E_\theta[n_1]}{n} = \theta$$

SOLUTION 2:
Compute $E[X^2]$: From $Var(X) = E[X^2] - E[X]^2$ it follows:

$$E[X^2] = Var(X) + E[X]^2 = kp(1-p) + k^2p^2$$

We define $\bar{X} = \sum_{i=1}^{n} X_i$ and $\bar{Y} = \sum_{i=1}^{n} X_i^2$, and we then set:

$$\bar{X} = E[X] = kp \;\text{ and }\; \bar{Y} = E[X^2] = kp(1-p) + k^2p^2$$

From the 1st equation it follows: $p = \frac{\bar{X}}{k}$. Plugging this into the 2nd equation:

$$\begin{aligned}
& \bar{Y} = k \cdot \frac{\bar{X}}{k} \cdot (1 - \frac{\bar{X}}{k}) + k^2 \left(\frac{\bar{X}}{k}\right)^2 \\
\Leftrightarrow \quad & \bar{Y} = \bar{X} \cdot (1 - \frac{\bar{X}}{k}) + \bar{X}^2 \\
\Leftrightarrow \quad & \bar{Y} = \bar{X} - \frac{\bar{X}^2}{k} + \bar{X}^2 \\
\Leftrightarrow \quad & \bar{Y} - \bar{X} - \bar{X}^2 = -\frac{\bar{X}^2}{k} \\
\Leftrightarrow \quad & \frac{\bar{X}^2}{\bar{X} + \bar{X}^2 - \bar{Y}} = k
\end{aligned}$$

In summary, we thus have:

$$\hat{k}_{MOM} = \frac{\bar{X}^2}{\bar{X} + \bar{X}^2 - \bar{Y}} \;\text{ and }\; \hat{p}_{MOM} = \frac{\bar{X}}{\hat{k}_{MOM}}$$

SOLUTION 3:
(a) The joint density is:

$$p(x_1, x_2, x_3|\theta) = \prod_{i=1}^{3} p(x_i|\theta) = e^{-3\lambda} \frac{\lambda^{x_1+x_2+x_3}}{x_1!x_2!x_3!} = h(x_1 + x_2 + x_3|\lambda) \cdot g(x1, x2, x3)$$

where

$$h(x_1 + x_2 + x_3|\lambda) = e^{-3\lambda}\lambda^{x_1+x_2+x_3} \text{ and } g(x1, x2, x3) = \frac{1}{x_1!x_2!x_3!}$$

It follows (factorization theorem) that $T(X_1, X_2, X_3) = X_1 + X_2 + X_3$ is sufficient statistic.

(b) Compute log likelihood (for $n = 1$):

$$l(\lambda) = \log(p(x_1|\lambda)) = -\lambda + x_1 \log(\lambda) - log(x_1!)$$

Take 1st and 2nd derivative:

$$\begin{aligned}
l'(\lambda) &= -1 + \frac{x_1}{\lambda} \\
l''(\lambda) &= -\frac{x_1}{\lambda^2}
\end{aligned}$$

The Fisher information (of a sample of size 1) is:

$$I(\lambda) = -E[l''(\lambda)] = \frac{E[X_1]}{\lambda^2} = \frac{1}{\lambda}$$

(c) We compute:

$$\begin{aligned} E[\hat{\lambda}_\star] &= E[\frac{X_1 + 2X_2 + 3X_3}{6}] = \frac{E[X_1] + 2E[X_2] + 3E[X_3]}{6}] = \frac{6\lambda}{6} = \lambda \\ Var(\hat{\lambda}_\star) &= Var(\frac{X_1 + 2X_2 + 3X_3}{6}) = \frac{Var(X_1) + 4Var(X_2) + 9Var(X_3)}{36}] = \frac{14\lambda}{36} \end{aligned}$$

The Cramer-Rao bound is $\frac{1}{n \cdot I(\lambda)} = \frac{\lambda}{3}$ and as $\frac{\lambda}{3} < \frac{14\lambda}{36}$, the estimator does not attain the Cramer-Rao bound.

(d) We compute the new estimator:

$$\begin{aligned} \hat{\lambda}_\star &= E[\hat{\lambda}_\diamond | X_1 + X_2 + X_3] = E[\frac{X_1 + 2X_2 + 3X_3}{6} | X_1 + X_2 + X_3] \\ &= \frac{1}{6}(E[X_1 | X_1 + X_2 + X_3] + 2E[X_2 | X_1 + X_2 + X_3] + 3E[X_3 | X_1 + X_2 + X_3]) \end{aligned}$$

As X_1, X_2, X_3 is an i.i.d. sample we have $E[X_i | X_1 + X_2 + X_3] = \frac{1}{3}(X_1 + X_2 + X_3)$, so that

$$\hat{\lambda}_\star = \frac{1}{6}(6 \cdot \frac{1}{3}(X_1 + X_2 + X_3)) = \frac{X_1 + X_2 + X_3}{3} = \bar{X}.$$

As $Var(\hat{\lambda}_\star) = \frac{1}{3} Var(X_1) = \frac{\lambda}{3}$, the new estimator attains the Cramer-Rao bound.

(e) Compute the joint density ratio:

$$\begin{aligned} W(X_1, X_2, X_3) &= \frac{p(X_1, X_2, X_3 | \lambda = 1)}{p(X_1, X_2, X_3 | \lambda = 3)} \\ &= \frac{e^{-3} \cdot \frac{1^{(\sum_{i=1}^3 X_i)}}{\prod_{i=1}^3 X_i!}}{e^{-9} \cdot \frac{3^{(\sum_{i=1}^3 x_i)}}{\prod_{i=1}^3 X_i!}} = \frac{e^{-3}}{e^{-9} \cdot 3^{(\sum_{i=1}^3 X_i)}} = e^6 \cdot \left(\frac{1}{3}\right)^{\sum_{i=1}^3 X_i} \end{aligned}$$

and we reject H_0 if

$$\begin{aligned} e^6 \cdot \left(\frac{1}{3}\right)^{\sum_{i=1}^3 X_i} < k &\Leftrightarrow 6 - \log(3) \sum_{i=1}^3 X_i < \log(k) \\ &\Leftrightarrow \sum_{i=1}^3 X_i > \frac{6 - \log(k)}{\log(3)} \end{aligned}$$

Under H_0 the statistic $\sum_{i=1}^3 X_i$ has a Poisson distribution with parameter $1 + 1 + 1 = 3$. Therefore we want the right hand-side to correspond to the $1 - \alpha = 0.95$ quantile of a Poisson distribution with parameter 3. The decision rule is that we reject the null hypothesis H_0 when $\sum_{i=1}^3 X_i$ takes a value equal to or larger than $q_{\lambda,\alpha} = q_{3,0.95}$.

SOLUTION 4:

(a) The log likelihood is:

$$
\begin{aligned}
l(\sigma^2) = \log(L(\sigma^2)) &= \log\left(\prod_{i=1}^{n}\left(\frac{x_i}{\sigma^2}\cdot\exp\{-\frac{x_i^2}{2\sigma^2}\}\right)\right) \\
&= \sum_{i=1}^{n}\log\left(\frac{x_i}{\sigma^2}\right) - \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2} \\
&= \sum_{i=1}^{n}\log(x_i) - n\cdot\log(\sigma^2) - \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}
\end{aligned}
$$

Compute the 1st derivative and set to zero:

$$
l'(\sigma^2) = -\frac{n}{\sigma^2} + 2\frac{\sum_{i=1}^{n} x_i^2}{(2\sigma^2)^2} = -\frac{n}{\sigma^2} + \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^4} = 0 \Leftrightarrow -n + \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2} = 0 \Leftrightarrow \sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{2n}
$$

To check whether this is really a maximum, we compute the 2nd derivative::

$$
l''(\sigma^2) = \frac{n}{(\sigma^2)^2} - 4\sigma^2\frac{\sum_{i=1}^{n} x_i^2}{(2\sigma^4)^2} = \frac{n}{\sigma^4} - \frac{\sum_{i=1}^{n} x_i^2}{\sigma^6}
$$

When plugging in $\sigma_\star^2 = \frac{\sum_{i=1}^{n} x_i^2}{2n}$ we get:

$$
\frac{n}{\sigma_\star^4} - \frac{\sum_{i=1}^{n} x_i^2}{\sigma_\star^6} = \frac{n}{\sigma_\star^4} - \frac{\sigma_\star^2\cdot 2n}{\sigma_\star^6} = -\frac{n}{\sigma_\star^4} < 0
$$

As this is negative, we indeed have a maximum, and $\hat{\sigma}_{ML}^2 = \frac{\sum_{i=1}^{n} x_i^2}{2n}$.

(b) We have:

$$
E[\hat{\sigma}_{ML}^2] = E\left[\frac{\sum_{i=1}^{n} X_i^2}{2n}\right] = \frac{1}{2n}\sum_{i=1}^{n} E[X_i^2]
$$

We note that $X_i = \sqrt{U_i^2 + W_i^2}$, so that $X_i^2 = U_i^2 + W_i^2$ and $\left(\frac{X_i}{\sigma}\right)^2 = \left(\frac{U_i}{\sigma}\right)^2 + \left(\frac{W_i}{\sigma}\right)^2$. As $\frac{U_i}{\sigma}$ and $\frac{W_i}{\sigma}$ are standard Gaussian, the square sum of both is $\chi^2(2)$ distributed and has expectation 2. Thus it follows:

$$
E\left[\left(\frac{X_i}{\sigma}\right)^2\right] = 2 \Leftrightarrow E[X_i^2] = 2\sigma^2
$$

So we have:

$$
E[\hat{\sigma}_{ML}^2] = \frac{1}{2n}\sum_{i=1}^{n} E[X_i^2] = \frac{1}{2n}\sum_{i=1}^{n} 2\sigma^2 = \frac{n2\sigma^2}{2n} = \sigma^2
$$

(c) We make use of the 2nd derivative with $n = 1$:

$$I(\sigma^2) = E[-l''(\sigma^2)] = E\left[-\frac{1}{\sigma^4} + \frac{X_1^2}{\sigma^6}\right] = -\frac{1}{\sigma^4} + \frac{E[X_1^2]}{\sigma^6} = -\frac{1}{\sigma^4} + \frac{1}{\sigma^4}E\left[\left(\frac{X_1}{\sigma}\right)^2\right]$$

Using a result from part **(b)**, namely that $E\left[\left(\frac{X_1}{\sigma}\right)^2\right] = 2$, we get:

$$I(\sigma^2) = -\frac{1}{\sigma^4} + \frac{2}{\sigma^4} = \frac{1}{\sigma^4}$$

(d) We compute the variance of the ML estimator:

$$Var(\hat{\sigma}^2_{ML}) = Var\left(\frac{\sum_{i=1}^{n} X_i^2}{2n}\right) = \frac{1}{4n^2}\sum_{i=1}^{n} Var(X_i^2)$$

Like in part (b), we use that $X_i = \sqrt{U_i^2 + W_i^2}$, so that $X_i^2 = U_i^2 + W_i^2$ and $\left(\frac{X_i}{\sigma}\right)^2 = \left(\frac{U_i}{\sigma}\right)^2 + \left(\frac{W_i}{\sigma}\right)^2$. As $\frac{U_i}{\sigma}$ and $\frac{W_i}{\sigma}$ are standard Gaussian, the square sum of both is $\chi^2(2)$ distributed and has variance 4. Thus it follows:

$$Var\left(\left(\frac{X_i}{\sigma}\right)^2\right) = 4 \Leftrightarrow Var(X_i^2) = 4\sigma^4$$

$$\text{hence:} \quad Var(\hat{\sigma}^2_{ML}) = \frac{1}{4n^2}\sum_{i=1}^{n} Var(X_i^2) = \frac{n \cdot 4\sigma^4}{4n^2} = \frac{\sigma^4}{n} = \frac{1}{nI(\sigma^2)}$$

so that the ML estimator attains the Cramer Rao bound.

(e) Asymptotically we have:

$$\begin{aligned} &\sqrt{n} \cdot (\hat{\sigma}^2_{ML} - \sigma^2) \sim \mathcal{N}(0, I(\sigma^2)^{-1}) \\ \Leftrightarrow \quad &\sqrt{I(\sigma^2)} \cdot \sqrt{n} \cdot (\hat{\sigma}^2_{ML} - \sigma^2) \sim \mathcal{N}(0, 1) \end{aligned}$$

We have $n = 9$, $\hat{\sigma}^2_{ML} = 4$ and the observed Fisher information is $I(4) = \frac{1}{16}$. This yields:

$$\begin{aligned} &\sqrt{\frac{1}{16}} \cdot \sqrt{9} \cdot (4 - \sigma^2) \sim \mathcal{N}(0, 1) \\ \Leftrightarrow \quad &\frac{3}{4}(4 - \sigma^2) \sim \mathcal{N}(0, 1) \end{aligned}$$

Let q_α be the α quantile of the $\mathcal{N}(0, 1)$, so that $q_\alpha = -q_{1-\alpha}$

$$P(q_{0.025} < \frac{3}{4}(4 - \sigma^2) < q_{0.975}) = 0.95 \Leftrightarrow P(4 + \frac{4}{3} \cdot (-q_{0.025} > \sigma^2 > 4 - \frac{4}{3} \cdot q_{0.025}) = 0.95$$

and the 0.95 CI is given by: $4 \pm \frac{4}{3} \cdot q_{0.025}$. With $q_{0.025} \approx -2$ we get the interval $[\frac{4}{3},\ \frac{20}{3}]$ or $[1.333,\ 6.667]$.

SOLUTION 5:

(a) Second order Taylor expansion:

$$\begin{aligned} l(\theta_0) &\approx l(\hat{\theta}_{ML}) + (\hat{\theta}_{ML} - \theta_0) \cdot l'(\hat{\theta}_{ML}) + \frac{1}{2}(\hat{\theta}_{ML} - \theta_0)^2 \cdot l''(\hat{\theta}_{ML}) \\ &\approx l(\hat{\theta}_{ML}) + \frac{1}{2}(\hat{\theta}_{ML} - \theta_0)^2 \cdot l''(\hat{\theta}_{ML}) \end{aligned}$$

as $l'(\hat{\theta}_{ML}) = 0$.

(b) We use the result from (a)

$$\begin{aligned} -2\log\left(\frac{L(\theta_0)}{\max\limits_{\theta\in\Theta}\{L(\theta)\}}\right) &= -2\log\left(\frac{L(\theta_0)}{L(\hat{\theta}_{ML})}\right) \\ &= -2 \cdot (l(\theta_0) - l(\hat{\theta}_{ML})) \\ &\approx -2 \cdot \left(l(\hat{\theta}_{ML}) + \frac{1}{2}(\hat{\theta}_{ML} - \theta_0)^2 \cdot l''(\hat{\theta}_{ML}) - l(\hat{\theta}_{ML})\right) \quad \textbf{see (a)} \\ &\approx -(\hat{\theta}_{ML} - \theta_0)^2 \cdot l''(\hat{\theta}_{ML}) \end{aligned}$$

(c) We recall:

$$I(\theta_0) := -E\left[\frac{d^2}{d\theta^2}\log(f(X_1|\theta))\right]|_{\theta=\theta_0}$$

We have:

$$\begin{aligned} \frac{1}{n} \cdot l''(\theta_0) &= \frac{1}{n} \cdot \left(\frac{d^2}{d\theta^2}\log(p(X_1, \dots, X_n|\theta))\right)|_{\theta=\theta_0} \\ &= \frac{1}{n} \cdot \left(\frac{d^2}{d\theta^2}\log(\prod_{i=1}^{n} p(X_i|\theta))\right)|_{\theta=\theta_0} \\ &= \frac{1}{n} \cdot \left(\frac{d^2}{d\theta^2}\sum_{i=1}^{n}\log(p(X_i|\theta))\right)|_{\theta=\theta_0} \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n}\left(\frac{d^2}{d\theta^2}\log(p(X_i|\theta))\right)|_{\theta=\theta_0} \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n}\frac{d^2 l_{X_i}}{d\theta^2}(\theta_0) \end{aligned}$$

and according to the Law of the Large Numbers (LLN) the later expression converges to

$$E\left[\frac{d^2 l_{X_1}}{d\theta^2}(\theta_0)\right] = E\left[\frac{d^2}{d\theta^2}\log(f(X_1|\theta))\right]|_{\theta=\theta_0} = -I(\theta_0)$$

END OF SOLUTIONS